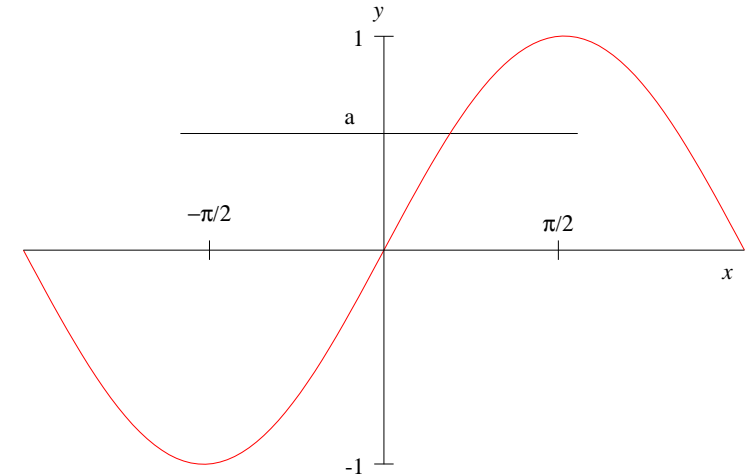


The probability density function of a sinusoid (1)



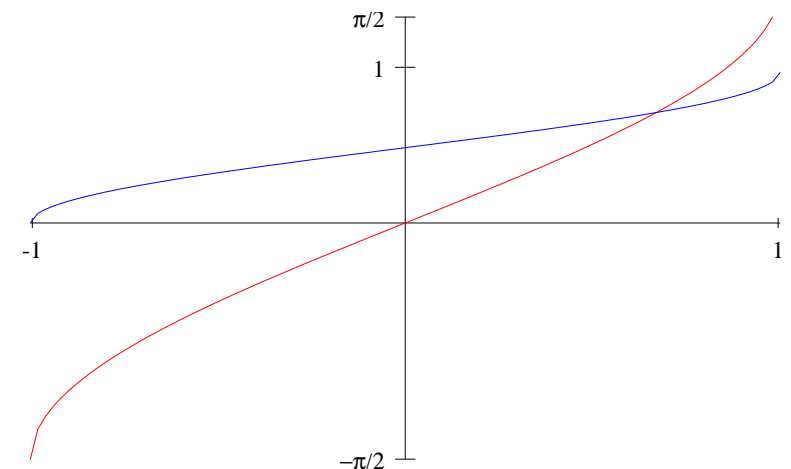
Let x be uniformly distributed over
so y is in $(-1,1)$



The cumulative probability $P(y \leq a) = P(\sin x \leq a)$, so, $P(x \leq \text{asin } a)$

Now normalize P so that $P(-1) = 0$ and $P(1) = 1$:

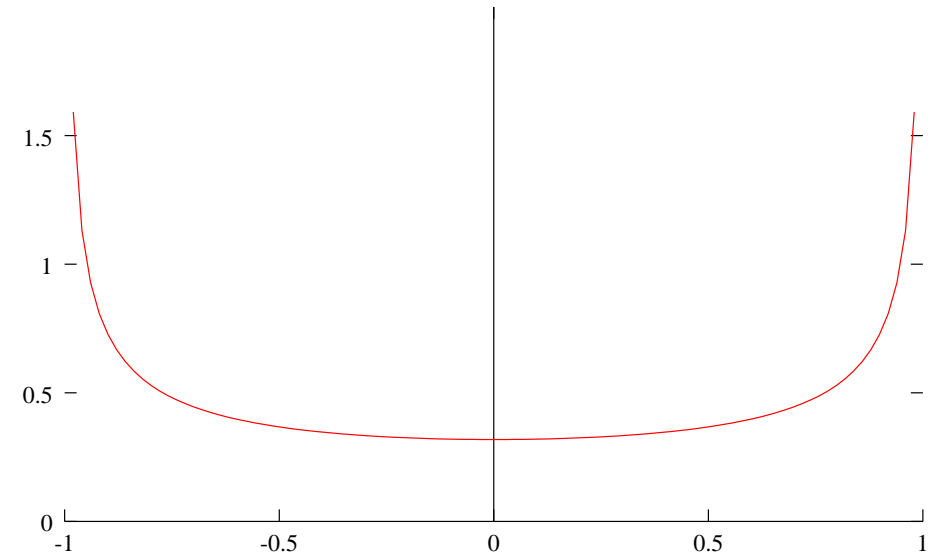
$$P(x \leq \text{asin } a) = \frac{\text{asin } a + \frac{\pi}{2}}{\pi} = \frac{\text{asin } a}{\pi} + \frac{1}{2}$$





Differentiate $P(a)$ to obtain the probability density function $p(a)$:

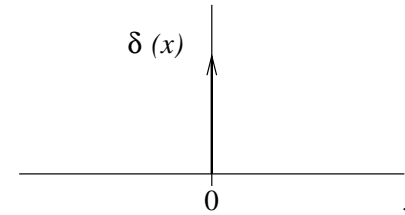
$$p(a) = \frac{dP(a)}{da} = \frac{1}{\pi \sqrt{(1-x^2)}}$$



The Dirac δ



$\delta(x)=0$ everywhere except at $x=0$



$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

$\delta(x)$ has unit surface

$$\int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x) \delta(x - x_0) dx = f(x_0)$$

It is a sampling function

$$\int_{-\infty}^{\infty} \delta(x) e^{-j2\pi fx} dx = 1$$

Fourier transform of $\delta(x)$ is unity

$$\int_{-\infty}^{\infty} 1 \cdot e^{j2\pi f x} df = \delta(x)$$

Inverse Fourier transform of unity is $\delta(x)$



$$g(t)*h(t)=\text{Fourier}^{-1}\{G(f)\cdot H(f)\}$$

$$G(f)=\text{Fourier}\{g(t)\}=\int_{-\infty}^{\infty}g(t)e^{-j2\pi ft}dt \quad (\text{Replace } t \text{ by } x)$$

$$H(f)=\text{Fourier}\{h(t)\}=\int_{-\infty}^{\infty}h(t)e^{-j2\pi ft}dt \quad (\text{Replace } t \text{ by } y)$$

$$g(t)*h(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x)h(y)e^{j2\pi f(t-x-y)}dxdydf \quad \text{Integrate over } f$$

$$g(t)*h(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x)h(y)\delta(t-x-y)dxdy \quad \text{Integrate over } x \text{ or } y$$

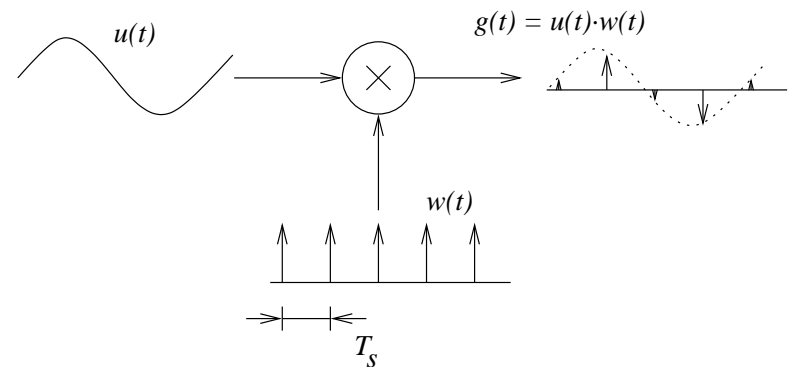
$$g(t)*h(t)=\int_{-\infty}^{\infty}g(t-y)h(y)dy$$

$$g(t)*h(t)=\int_{-\infty}^{\infty}g(x)h(t-x)dx$$

} The convolution integral



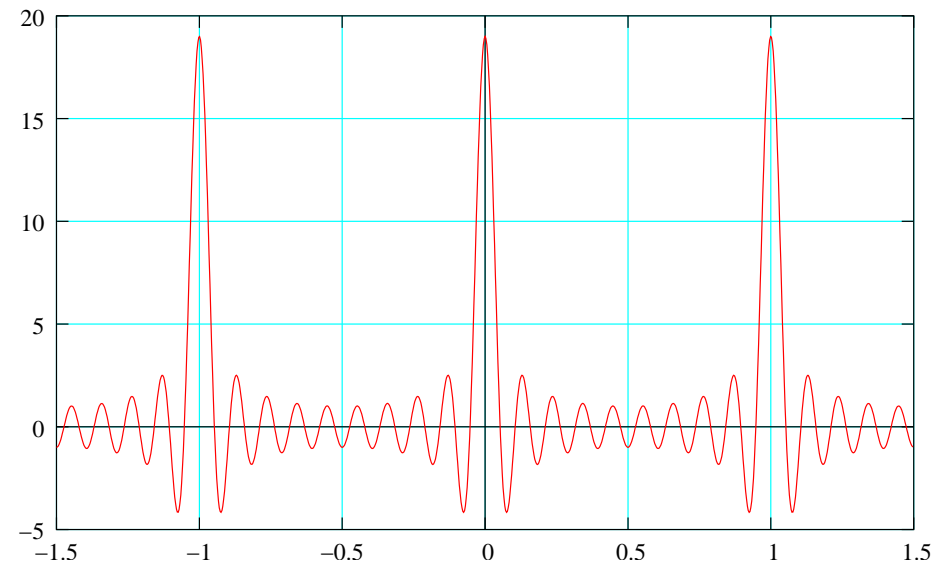
$$g(t) = u(t) \cdot w(t)$$



First the Fourier transform of $w(t)$:

$$W(f) = \text{Fourier} \{ w(t) \} = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - nT_s) e^{-j2\pi ft} dt$$

$$W(f) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t - nT_s) e^{-j2\pi ft} dt$$



$$W(f) = \sum_{n=-\infty}^{\infty} e^{-j2\pi f n T_s} = 1 + 2 \sum_{n=1}^{\infty} \cos 2\pi f n T_s = \sum_{n=-\infty}^{\infty} \delta(f - n F_s)$$



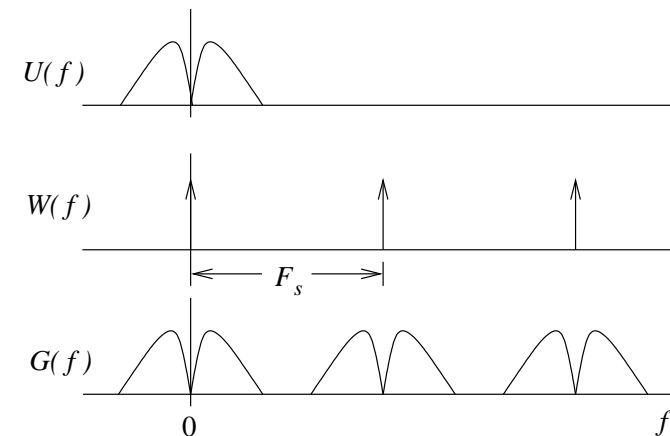
Convolve spectrum of sampling waveform with spectrum of input signal:

$$G(f) = U(f) * W(f) = \int_{-\infty}^{\infty} U(\phi) W(f - \phi) d\phi$$

$$G(f) = \int_{-\infty}^{\infty} U(\phi) \sum_{n=-\infty}^{\infty} \delta(f - nF_s - \phi) d\phi$$

$$G(f) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} U(\phi) \delta(f - nF_s - \phi) d\phi$$

$$G(f) = \sum_{n=-\infty}^{\infty} U(f - nF_s)$$





Clock signal: $V(t) = \sin(2\pi F_s t + \varphi(t))$

Significant instants: $2\pi F_s t_1 + \varphi(t_1) = 0$ and $2\pi F_s t_2 + \varphi(t_2) = 2\pi N$

$$2\pi F_s (t_2 - t_1) + \varphi(t_2) - \varphi(t_1) = 2\pi N$$

The time between these instants is an integer number of periods plus some jitter:

$$t_2 - t_1 = \frac{N}{F_s} + \Delta t$$

$$2\pi F_s \left(\frac{N}{F_s} + \Delta t \right) + \varphi(t_2) - \varphi(t_1) = 2\pi N$$

$$\Delta t = \frac{1}{2\pi F_s} (\varphi(t_1) - \varphi(t_2))$$



Expected value of variance:

$$\langle \Delta t^2 \rangle = \frac{1}{4\pi^2 F_s^2} (\langle \varphi(t_1)^2 \rangle - 2\langle \varphi(t_1)\varphi(t_2) \rangle + \langle \varphi(t_2)^2 \rangle)$$

Jitter does not depend on chosen instant:

$$\langle \varphi(t_1)^2 \rangle = \langle \varphi(t_2)^2 \rangle = \langle \varphi(t)^2 \rangle = \int_0^{\infty} S_{\varphi}(f) df$$

Variance of middle term is a covariance: Use cosine transform

$$\langle \varphi(t_1)\varphi(t_2) \rangle = \int_0^{\infty} S_{\varphi}(f) \cos(2\pi f \tau) df \quad \text{with} \quad \tau = t_1 - t_2$$



$$\langle \Delta t^2 \rangle = \frac{1}{2\pi^2 F_s^2} \int_0^{\infty} S_{\varphi}(f) (1 - \cos(2\pi f \tau)) df$$

$$\langle \Delta t^2 \rangle = \frac{1}{\pi^2 F_s^2} \int_0^{\infty} S_{\varphi}(f) (\sin^2(\pi f \tau)) df$$

RMS jitter

$$\langle \Delta t \rangle = \frac{1}{\pi F_s} \sqrt{\int_0^{\infty} S_{\varphi}(f) (\sin^2(\pi f \tau)) df}$$